Phys 410
Fall 2015
Lecture #21 Summary
10 November, 2015

A surprising result is that any object, no matter how irregular, always has 3 perpendicular principal axes (for a given choice of origin), for which the angular momentum vector and angular velocity are parallel. In other words, for any object we can find three perpendicular axes around which the object will rotate without "wobbling". The formal statement is this: For any rigid body and any choice of origin O there are three mutually perpendicular principal axes through O. This amounts to finding three perpendicular axes through O for which the calculation of the inertia tensor using these coordinate axes yields a diagonal matrix. This result arises from the linear algebraic properties of any real symmetric matrix (namely $\bar{\bar{I}}$) – it can always be diagonalized and the eigenvalues are real.

How to find the principal axes of an arbitrary object? We are looking for three directions for the angular velocity vector $\vec{\omega}$ to create an angular momentum vector that satisfies $\vec{L} = \lambda\vec{\omega}$, where λ is some real scalar number. This is the condition for two vectors to be parallel. Since in addition we know that in general $\vec{L} = \bar{\bar{I}}\vec{\omega}$, we can combine these two equations to find: $\bar{\bar{I}}\vec{\omega} = \lambda\vec{\omega}$, which is a classic eigenvalue problem. This equation states that a matrix multiplying a vector produces the same vector multiplied by a real number, the eigenvalue. The eigenvectors of this equation constitute the angular velocity directions that diagonalize the intertia tensor, and constitute the principal axes. These three vectors span the 3-dimensional coordinate space and are therefore mutually perpendicular.

We write $\lambda\vec{\omega} = \lambda\bar{\bar{1}}\vec{\omega}$, where $\bar{\bar{1}}$ is the 3x3 unit matrix, and then construct the eigenvalue matrix equation: $\left(\bar{\bar{I}} - \lambda\bar{\bar{1}}\right)\vec{\omega} = 0$. The only way to get non-trivial solutions from this equation is to demand that $det\left(\bar{\bar{I}} - \lambda\bar{\bar{1}}\right) = 0$. This yields three eigenvlaues and three eigen-functions. We examined the case of the cube rotated on an axis that passes through one corner of the cube, for which we calculate the inertia tensor above. This inertia tensor yields a characteristic equation $det\left(\bar{\bar{I}} - \lambda\bar{\bar{1}}\right) = (2\mu - \lambda)(11\mu - \lambda)^2 = 0$, where $\mu = Ma^2/12$, giving $\lambda = 2\mu$ as an eigenvalue and $\lambda = 11\mu$ as a double eigenvalue. The eigenvector associated with $\lambda_1 = 2\mu$ is $\widehat{\omega_1} = \frac{1}{\sqrt{3}}(1,1,1)$, which represents the body diagonal of the cube. The cube has a principal moment of inertia of $2\mu = Ma^2/6$ for rotation about this axis. The other two eigenvalues yield only the condition $\omega_x + \omega_y + \omega_z = 0$ on the eigenvectors, which simply mean that they have to be perpendicular to $\widehat{\omega_1}$. We are free to choose any two such directions that are mutually perpendicular. A set of possible choices are $\widehat{\omega_2} = \frac{1}{\sqrt{6}}(2,-1,-1)$, and $\widehat{\omega_3} = \frac{1}{\sqrt{2}}(0,1,-1)$, for which the cube has moment of inertia $11\mu =$

$11Ma^2/12$. To summarize, the principal axes $\widehat{\omega_1}$, $\widehat{\omega_2}$, $\widehat{\omega_3}$ diagonalize the inertia tensor as $\bar{\bar{I}} = \frac{Ma^2}{12}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}$.

We considered the motion of a "top" or gyroscope that was set into motion at angular velocity ω along one of its principal axes and then supported at a single point on its rotation axis. The top is rotating about one of its principal axes, which we will call the 3-axis, with direction $\widehat{e_3}$. The top is observed to precess in a cone around the vertical direction $\hat{z}$. We can write the angular momentum as $\vec{L} = \lambda_3 \omega \widehat{e_3}$, where λ_3 is the principal moment for this axis. There are two forces acting on the top, the normal force at the point of support, and the weight, acting on the center of mass. We take the origin to be at the point of support so that only the weight exerts a torque. The torque leads to a time rate of change of the angular momentum: $\vec{\Gamma} = \dot{\vec{L}}$. The torque is $\vec{\Gamma} = \vec{R} \times M\vec{g}$, which points in a direction perpendicular to $\widehat{e_3}$, and therefore $\vec{L}$. This means that $|\vec{L}|$ remains fixed (hence ω is constant), but the direction of $\vec{L}$ will change. We found that $\dot{\widehat{e_3}} = \vec{\Omega} \times \widehat{e_3}$, where $\vec{\Omega} = \frac{RMg}{\lambda_3 \omega}\hat{z}$, showing that the principal axis of the top $\widehat{e_3}$ is rotating around the $\hat{z}$ axis at angular velocity $\frac{RMg}{\lambda_3 \omega}$. This is the rate of precession. From the demonstration we saw that as the gyroscope winds down (ω decreases), the rate of precession increases, consistent with this result.

We then considered the description of Newton's second law from the perspective of an observer on the rotating object. The observer in the "body frame" can identify the principal axes of the object and describe the angular momentum using the diagonalized inertia tensor as $\vec{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$. An inertial observer in the "space frame" is in position to identify correctly the net torque $\vec{\Gamma}$ acting on the angular momentum vector, and to write Newton's second law of motion (in rotational form) as $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{space}$. We learned how to translate the time derivative of a vector quantity from an inertial frame to a rotating reference frame in Lecture 10: $\left(\frac{d\vec{Q}}{dt}\right)_{Space} = \left(\frac{d\vec{Q}}{dt}\right)_{Body} + \vec{\Omega} \times \vec{Q}$, where $\vec{Q}$ is the vector in question and the non-inertial reference frame is rotating with angular velocity $\vec{\Omega}$. In this case we can write the equations of motion as witnessed in the body frame as $\vec{\Gamma} = \left(\frac{d\vec{L}}{dt}\right)_{Body} + \vec{\omega} \times \vec{L}$, which translates in component form into the Euler equations:

$$\Gamma_1 = \lambda_1\dot{\omega}_1 - \omega_2\omega_3(\lambda_2 - \lambda_3)$$

$$\Gamma_2 = \lambda_2\dot{\omega}_2 - \omega_1\omega_3(\lambda_3 - \lambda_1)$$

$$\Gamma_3 = \lambda_3\dot{\omega}_3 - \omega_1\omega_2(\lambda_1 - \lambda_2)$$

This set of equations describes how the angular velocity vector evolves as it is acted upon by a net external torque. The hard part of using these equations is taking the torque from the space frame and expressing it in component form in the body frame (i.e. Γ_1, Γ_2, Γ_3). When applied to the case of the spinning top discussed above, we note that $\Gamma_3 = 0$ (the torque acts in a direction perpendicular to $\widehat{e_3}$) and $\lambda_1 = \lambda_2$, hence $\lambda_3 \dot{\omega}_3 = 0$, so that ω_3 is constant. Thus the angular velocity vector remains aligned with 3-axis and no other component of $\vec{\omega}$ is excited.